Comments on Theoretical Problems in Nonsymmetric Gravitational Theory

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Abstract

Damour, Deser and McCarthy have claimed that the nonsymmetric gravitational theory (NGT) is untenable due to curvature coupled ghost modes and bad asymptotic behavior. This claim is false for it is based on a physically inaccurate treatment of wave propagation on a curved background and an incorrect method for extracting asymptotic behavior. We show that the flux of gravitational radiation in NGT is finite in magnitude and positive in sign.

The nonsymmetric gravitational theory (NGT) has been extensively studied over a period of years[1,2] and these studies have shown that the theory is a mathematically consistent alternative to Einstein's gravitational theory (EGT). Other possible versions of nonsymmetric gravitational theories [3,4] have either been shown to possess ghost poles in the linear approximation or not to contain static spherically symmetric solutions, which have Schwarzschild-like behavior at large distances, unless the parameter describing the Schwarzschild mass is forced to be negative definite[4].

The NGT Lagrangian without sources is of the form[2]:

$$\mathcal{L}_{NGT} = \sqrt{-g}g^{\mu\nu}R_{\mu\nu}(W) = \sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma) + \frac{2}{3}(\sqrt{-g}g^{[\nu\mu]})_{,\nu}W_\mu, \tag{1}$$

where

$$W^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \frac{2}{3}\delta^{\lambda}_{\mu} W_{\nu}, \quad W_{\nu} = \frac{1}{2}(W^{\lambda}_{\nu\lambda} - W^{\lambda}_{\lambda\nu}) = W^{\lambda}_{[\nu\lambda]}. \tag{2}$$

The empty space field equations which follow from (1) are

$$R_{(\mu\nu)}(\Gamma) = 0, \tag{3}$$

$$R_{[\mu\nu]}(\Gamma) = \frac{2}{3} W_{[\nu,\mu]}, \tag{4}$$

$$g_{\mu\nu,\lambda} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} - g_{\mu\alpha}\Gamma^{\alpha}_{\lambda\nu} = 0, \tag{5}$$

$$(\sqrt{-g} g^{[\mu\nu]})_{,\nu} = 0. \tag{6}$$

These field equations must represent 12 independent equations for the 12 independent field variables $g_{\mu\nu}$ (there exist four arbitrary coordinate transformations: $x'^{\mu} = (\partial x'^{\mu}/\partial x^{\alpha})x^{\alpha}$, which can be used to remove 4 of the 16 $g_{\mu\nu}$'s).

Eq.(4) can be decomposed into the two sets of equations:

$$R_{\{[\mu\nu];\sigma\}}(\Gamma) \equiv R_{[\mu\nu]}(\Gamma)_{,\sigma} + R_{[\nu\sigma]}(\Gamma)_{,\mu} + R_{[\sigma\mu]}(\Gamma)_{,\nu} = 0, \tag{7}$$

and

$$R_{[\mu\nu]}(\Gamma)^{;\nu} = \frac{2}{3} W_{[\nu,\mu]}{}^{;\nu}, \tag{8}$$

where ; denotes covariant differentiation with respect to the connection $\Gamma^{\lambda}_{\mu\nu}$. Eqs.(6) and (7) are constrained, in turn, by the identities

$$\left(\sqrt{-g} g^{[\mu\nu]}\right)_{,\nu,\mu} = 0, \tag{9}$$

$$\epsilon^{\mu\nu\sigma\rho} R_{\{[\mu\nu],\sigma\}}(\Gamma)_{,\rho} = 0. \tag{10}$$

The field equations are further constrained by the four Bianchi identities,

$$[\sqrt{-g} g^{\alpha\nu} G_{\rho\nu}(\Gamma) + \sqrt{-g} g^{\nu\alpha} G_{\nu\rho}(\Gamma)]_{,\alpha} + [\sqrt{-g} g^{\mu\nu}]_{,\rho} G_{\mu\nu}(\Gamma) = 0, \tag{11}$$

where $G_{\mu\nu}(\Gamma) = R_{\mu\nu}(\Gamma) - 1/2 g_{\mu\nu} R(\Gamma)$.

Employing Eq.(5) to eliminate Γ in favor of $g_{\mu\nu}$, Eqs.(3), (6) and (7) represent 18 equations for $g_{\mu\nu}$. Taking into account the six identites (9), (10) and (11), this latter set of equations provides 12 independent field equations for the 12 independent field variables, $g_{\mu\nu}$. At no stage have we had to refer to the vector W_μ. W_μ *does not describe dynamical degrees of freedom*, in keeping with the fact that it corresponds to a Lagrange multiplier. Of course, one could use Eq.(8) to solve for W_μ in terms of the previously determined $g_{\mu\nu}$ but this would serve no useful purpose.

The field equations (3), (6) and (7) yield the following static spherically symmetric solution:

$$ds^2 = \left(1 - \frac{2m}{r}\right)\left(1 + \frac{\ell^4}{r^4}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{12}$$

and

$$g_{[10]}(r) = \frac{\ell^2}{r^2}. \tag{13}$$

Here, m and ℓ^2 are the two constants of integration identified with the mass and the NGT source parameter. Thus, in NGT there are now two sources of the *pure* gravitational field. We see that the source parameter ℓ^2 enters into the theory in a *nontrivial way*, since ℓ^2 and m couple nonlinearly in g_{00}. The parameter ℓ^2 has been identified at a phenomenological microscopic level with the conserved particle number [2].

In a weak field approximation obtained from expanding $g_{\mu\nu}$ about the Minkowski spacetime metric, $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{14}$$

where $\epsilon \ll 1$, the field equations take the form to lowest order:

$$\Box\, h_{(\mu\nu)} - h_{(\nu\sigma),\mu}{}^{,\sigma} - h_{(\mu\sigma),\nu}{}^{,\sigma} + h_{,\mu,\nu} = 0, \tag{15}$$

$$h_{[\mu\beta]}{}^{,\beta} = 0, \tag{16}$$

$$\Box\, h_{[\mu\nu]} = \frac{4}{3} W_{[\nu,\mu]}, \tag{17}$$

where $\Box = \partial^\mu \partial_\mu$ and $h = \eta^{\alpha\beta} h_{\alpha\beta}$. We see that the symmetric part of the field equations decouples from the skew part, and that it is identical to that of EGT. The skew equations take the form of Kalb-Ramond-Kimura equations[5] in a permanently fixed gauge. The spin of $h_{[\mu\nu]}$ is $J^P = 0^+$ and it is not difficult to show *that there are no ghost poles* due to the existence of a restricted gauge invariance[6,7]:

$$\delta h_{[\mu\nu]} = \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu}, \quad \Box\, \epsilon_\mu - \epsilon_{\nu,\mu}{}^{,\nu} = 0. \tag{18}$$

Let us define the longitudinal and transverse projection operators:

$$P^L_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\Box}, \quad P^T_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}, \tag{19}$$

then we find that

$$h^{TT}_{[\mu\nu]} = P^{T\alpha}_\mu P^{T\beta}_\nu h_{[\alpha\beta]}, \tag{20}$$

$$h^{LL}_{[\mu\nu]} = P^{L\alpha}_\mu P^{L\beta}_\nu h_{[\alpha\beta]} = 0, \tag{21}$$

$$h^{LT}_{[\mu\nu]} = P^{L\alpha}_{[\mu} P^{T\beta}_{\nu]} h_{[\alpha\beta]}. \tag{22}$$

Also, using the gauge condition $\partial^\alpha W_\alpha = 0$, we have

$$W^{TT}_{[\mu,\nu]} = P^{T\alpha}_\mu P^{T\beta}_\nu W_{[\alpha,\beta]} = 0. \tag{23}$$

Thus, the field equations to linear order are:

$$h^{LT}_{[\mu\beta]}{}^{,\beta} = 0, \tag{24}$$

$$\Box\, h^{TT}_{[\mu\nu]} = 0, \tag{25}$$

and

$$\Box h^{LT}_{[\mu\nu]} = \frac{4}{3}W_{[\nu,\mu]}. \tag{26}$$

As in the case of the exact field equations, we see that Eqs.(24) and (25) completely determine the $h_{[\mu\nu]}$ without reference to the Lagrange multiplier W_μ.

Kelly[4] and Damour, Deser and McCarthy (DDM)[8] have proposed expanding $g_{\mu\nu}$ about a pure Einstein local vacuum background metric:

$$g_{\mu\nu} = g_{E(\mu\nu)} + \epsilon h_{\mu\nu}, \tag{27}$$

where $g_{E(\mu\nu)}$ denotes the background Einstein metric. They work at the level of the field equations, keeping only the first order in $h_{[\mu\nu]}$ and all orders in $g_{E(\mu\nu)}$. The resulting field equations are[8]:

$$\bar{R}_{\mu\nu}(g_E) = 0, \tag{28}$$

$$\bar{D}^\alpha F_{\mu\nu\alpha} - 4\bar{R}^{\alpha\ \beta}_{\ \mu\ \nu}(g_E)h_{[\alpha\beta]} = \frac{4}{3}W_{[\nu,\mu]}, \tag{29}$$

$$\bar{D}^\nu h_{[\mu\nu]} = 0, \tag{30}$$

where $F_{\mu\nu\alpha}$ and $\bar{D}^\alpha$ denote the cyclic curl of $h_{[\mu\nu]}$ and the background covariant derivative, respectively. All operations are in the background metric g_E space.

As before, the equations (28)-(30) can be solved for $h_{[\mu\nu]}$ without specifying the Lagrange multiplier W_μ. However, in the gauge $\partial^\alpha W_\alpha = 0$, DDM proceed to take the divergence of (29) which gives the wave equation:

$$\bar{D}^\mu \bar{D}_\mu W_\nu = -3\bar{D}^\mu(\bar{R}^{\alpha\ \beta}_{\ \mu\ \nu}(g_E)h_{[\alpha\beta]}). \tag{31}$$

They argue that this is an inhomogeneous wave equation for W_μ, so W_μ has $1/r$ fall off in the wave zone. They then go on to argue that inserting this information back into Eq.(29), drives $h_{[\mu\nu]}$ to have unsatisfactory asymptotic behaviour. This approach is incorrect. Firstly, the source term for this wave equation is not confined to the near zone (i.e. it is not compact), so one cannot extract a $1/r$ asymptotic form in the usual way, instead the equation must be solved globally. For example, the static spherically symmetric solution has $h_{[10]} = l^2/r^2$, $\bar{R}^{10}{}_{10}(g_E) = 2m/r^3$ which, when inserted into (29) or (31), gives $W_0 = 3ml^2/2r^4$, in agreement with the exact solution. Secondly, Eq.(31) corresponds to the redundant field equation (8) for the auxiliary field W_μ and so plays no part in determining the $h_{[\mu\nu]}$. The field equations that do in fact determine $h_{[\mu\nu]}$ are obtained by expanding Eqs.(6) and (7):

$$\bar{D}^\nu h_{[\mu\nu]} = 0, \tag{32}$$

$$\bar{D}^\alpha \bar{D}_\alpha F_{\mu\nu\sigma} - h_{[\kappa\{\mu]} \left(\bar{R}^{\kappa}_{\nu\sigma\}}{}^{\lambda}(g_E) \right)_{;\lambda} + \bar{R}^{\alpha\beta}_{\{\mu\nu}(g_E) h_{[\alpha\beta];\sigma\}} = 0. \tag{33}$$

DDM go on to assert that the second term in (29) couples the background curvature $\bar{R}$ to $h_{[\mu\nu]}$, causing a violation of the restricted gauge invariance and thereby producing ghost-like longitudinal modes. This assertion is false for Eqs. (28)-(30), as they stand, do not sensibly describe wave propagation. When studying gravitational waves propagating on a curved background, careful attention must be paid to the physical situation being modelled. A gravitational wave is a small ripple on the geometry of a curved but slowly varying background. The words "small, ripple and slowly varying" convey an obvious physical picture, which must be correctly modelled by the mathematics. In deriving Eq.(29) only the amplitude of the perturbation has been controlled by taking $\epsilon \ll 1$. Implicit in the linearisation is that the curvature induced by the perturbation can be neglected in comparison to the background curvature. Nothing has been done to enforce the geometrical optics condition that the background varies more slowly than the disturbance. All these conditions can be made concrete as follows. In terms of the decomposition (27), two characteristic lengths, L and λ, can be defined as the scales over which the background and the wave vary,

$$\partial g_{E(\mu\nu)} \sim \frac{g_{E(\mu\nu)}}{L}, \quad \partial h_{\mu\nu} \sim \frac{h_{\mu\nu}}{\lambda}. \tag{34}$$

The curvature induced by the wave is of order (ϵ^2/λ^2), while the background curvature is of order $(1/L^2)$. To be able to consistently neglect self-gravitation requires

$$\frac{\epsilon}{\lambda} \ll \frac{1}{L}. \tag{35}$$

Furthermore, to account for the distinction between the wave being a ripple and the background being slowly varying, we demand that

$$\delta = \frac{\lambda}{L} \ll 1, \tag{36}$$

so that we have the complete set of conditions:

$$\epsilon \ll \delta \ll 1. \tag{37}$$

One must always have $\delta \ll 1$ as well as $\epsilon \ll 1$ if the meaning of a gravitational wave is to make any sense![10]. If these conditions are not enforced, then the analysis is no longer in the realm of geometric optics and notions such as local gauge invariance become meaningless.

Indeed, the same is true in EGT when considering gravitational waves propagating on a curved background. It is found that the Lagrangian for the lowest order wave equation (without gauge conditions) is not invariant under infinitesimal gauge transformations, and it is only when the above conditions (37) are enforced that a conserved, gauge invariant energy-momentum tensor with positive definite flux in the wave zone is obtained[11].

Returning to NGT, and properly implementing the wave and background decomposition, we find from equation (29) that to order δ:

$$\bar{D}^{\alpha}\bar{D}_{\alpha}h_{[\mu\nu]} = \frac{4}{3}W_{[\nu,\mu]}. \tag{38}$$

This set of field equations together with (30) satisfy a restricted gauge invariance and there are no longitudinal ghost-like modes.

The total energy of a system, in NGT, is given by

$$E = \int t^{00} d^3x, \tag{39}$$

where $t^{\mu\nu}$ denotes the energy-momentum pseudo-tensor. From the conservation equations ${t^{\mu\nu}}_{,\nu} = 0$, one finds that for localized sources the total energy is conserved up to a flux of energy carried to infinity by gravitational waves. Thus, the rate of energy loss is given by

$$\frac{dE}{dt} = -R^2 \oint t^{0i} \hat{n}_i d\Omega, \tag{40}$$

where the integration is over a sphere of radius R in the wave zone, and $\hat{n}_i$ is an outward pointing unit vector. A calculation yields

$$\frac{dE}{dt} = -\frac{R^2}{32\pi} \oint [({h^{TT\,(ij)}}_{,0})^2 + ({h^{TT\,[ij]}}_{,0})^2] d\Omega. \tag{41}$$

In the work of Krisher[9], contributions from W_μ were erroneously kept in the radiation flux equation. These additional terms came from the combination $4/3\delta_{ik}(h^{[0i]}W^{[j,k]} + h^{[ij]}W^{[k,0]})$ in $t^{(0j)}$. Using equation (26) we see that this combination can be re-written as $\delta_{ik}(h^{[0i]} \Box h^{[kj]}_{LT} + h^{[ij]} \Box h^{[0k]}_{LT})$ which falls off at least as $1/R^3$ by dint of equation (24). Thus, we see that these terms should have been dropped along with all the other terms that fall off faster than $1/R^2$.

Using Krisher's solution for $h^{[ij]}_{TT}$, given by Eq.(4.39b) in his paper[9], a calculation shows that the second term in (41) vanishes, and the gravitational flux is determined just by the familiar Einstein quadrupole formula:

$$\left(\frac{dE}{dt}\right)_{\rm quad} = -\left\langle \left(\frac{\mu^2 m^2}{r^4}\right) \frac{8}{15}(12v^2 - 11\dot{r}^2) \right\rangle, \tag{42}$$

where m and μ are the total mass and the reduced mass of the system, respectively, $\vec{v}$ is the relative orbital velocity of gravitationally bound objects, $\dot{r} = dr/dt$ for the orbital separation r, and the angular brackets denote an average over an orbital period. Thus, there is no dipole radiation in NGT[6] and the flux of energy carried to infinity is positive definite. While DDM correctly point out the error in the signs of the skew terms in Krisher's radiation flux equation, the observation is inconsequential since none of the terms contribute to the flux.

The above arguements have now been supported by an *exact* axi-symmetric gravitational wave solution in NGT[12]. The skew metric terms $h_{[\mu\nu]}$ were found to fall off as $1/r^2$ while the auxiliary vector field W_μ falls off like $1/r^3$ or faster. Again, this clearly contradicts the assertions made by DDM about bad asymptotic behavior in NGT. The gravitational wave flux in the wave zone was found to be positive definite as in EGT.

In summary, the claim by DDM that NGT encounters problems with unphysical longitudinal modes and with bad asymptotic behavior is incorrect. The equations describing wave propagation in NGT about a Riemannian background are gauge invariant, when the usual geometric optics conditions are enforced. These are the same conditions that must be applied to obtain sensible results in EGT. DDM's claim that the skew metric components $h_{[\mu\nu]}$ fail to vanish asymptotically is based on an erroneous method for extracting the asymptotic behavior of the Lagrange multiplier W_μ. When the six field equations for the six components of $h_{[\mu\nu]}$ are solved (which naturally do not refer to the Lagrange multiplier), it is found that NGT has healthy asymptotic behavior. A direct calculation of the gravitational energy flux in NGT for a binary system shows the flux to be finite in magnitude and positive in sign.

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